# Online Appendix to
# Monotone Ecological Inference

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# A Formal Description of Ecological Regression and Neighborhood Model Estimators

In the main text, we focus on the estimands associated with the Ecological Regression and Neighborhood Model estimators without describing how their sample analogues are constructed. Here, we formally describe how these estimators are constructed using the type of data a researcher would observe in EI settings.

### Ecological Regression Estimators

The ecological regression estimator for $D$ is given by

$$\widehat{D}_{ER} = \frac{\sum\limits_{n \in \mathcal{N}} P_n \, \widetilde{Y}_n \, \widetilde{X}_n}{\sum\limits_{n \in \mathcal{N}} P_n \left( \widetilde{X}_n \right)^2}$$

where $\widetilde{X}_n = X_n - \frac{\sum_n P_n X_n}{\sum_n P_n}$ denotes the demeaned value of $X_n$, and similarly for $\widetilde{Y}_n = Y_n - \frac{\sum_n P_n Y_n}{\sum_n P_n}$.

The ecological regression estimators for the group-level means are correspondingly given by[22]

$$\widehat{Y}^0_{ER} = \sum_{n \in \mathcal{N}} P_n \, Y_n - \left( \frac{\sum\limits_{n \in \mathcal{N}} P_n \, \widetilde{Y}_n \, \widetilde{X}_n}{\sum\limits_{n \in \mathcal{N}} P_n \, (\widetilde{X}_n)^2} \right) \left( \sum_{n \in \mathcal{N}} P_n \, X_n \right)$$

and

$$\widehat{Y}^1_{ER} = \sum_{n \in \mathcal{N}} P_n \, Y_n + \left( \frac{\sum\limits_{n \in \mathcal{N}} P_n \, \widetilde{Y}_n \, \widetilde{X}_n}{\sum\limits_{n \in \mathcal{N}} P_n \, (\widetilde{X}_n)^2} \right) \left( \sum_{n \in \mathcal{N}} P_n \, (1 - X_n) \right)$$

### The Neighborhood Model Estimators

The neighborhood model estimators for the group-level means are weighted averages of the neighborhood-level outcomes, with weights given by the

[22] One can equivalently define the ecological regression estimates for the group means as the estimated coefficients of the weighted regression of $Y_N$ on $X_N$ and $1 - X_N$.

group's prevalence in the neighborhood:

$$\widehat{Y}^1_{NM} = \frac{\sum\limits_{n\in\mathcal{N}} P_n X_n Y_n}{\sum\limits_{n\in\mathcal{N}} P_n X_n}$$

and

$$\widehat{Y}^0_{NM} = \frac{\sum\limits_{n\in\mathcal{N}} P_n \left(1 - X_n\right) Y_n}{\sum\limits_{n\in\mathcal{N}} P_n \left(1 - X_n\right)}$$

The neighborhood model estimator for the difference in group means, $\widehat{D}_{NM}$, is correspondingly defined as

$$\widehat{D}_{NM} = \frac{\sum\limits_{n\in\mathcal{N}} P_n X_n Y_n}{\sum\limits_{n\in\mathcal{N}} P_n X_n} - \frac{\sum\limits_{n\in\mathcal{N}} P_n (1 - X_n) Y_n}{\sum\limits_{n\in\mathcal{N}} P_n (1 - X_n)}$$

# B Generalizations

This Appendix Section considers the applicability of our results to EI settings in which either the outcome ($Y$) or group membership ($X$) is non-binary.

## B.1 Non-Binary Outcomes

In the main text we assumed that individual outcomes ($Y$) were binary. In some settings it could be that $Y$ is non-binary but discrete, or continuous with $Y \in \mathbb{R}$.

As long as the conditional expectation of $Y$ is well-defined and bounded for each $n$ and $x$, most of our results go through unmodified. The main difference is that the parameters that define the MOB interval now take on a somewhat more complicated form (see the statements of Propositions C.1and C.2 in Appendix Section C). In addition, to the difference in the definition of the MOB interval, our results that characterize the width of the MOB interval are affected by the possibility that $Y$ is non-binary, with Proposition 5 no longer holding in general (see Proposition C.3).

Appendix Section C proves all of our results under the assumption that $Y$ is non-binary, and where applicable, shows how the Propositions in the main text follow as a special case when a binary outcome is assumed.

## B.2 Non-Binary Group Membership

Our discussion has assumed that group membership is binary. Suppose instead that there are $G+1$ discrete groups, indexed by $g \in \{0, 1, ..., G\}$. To identify $\mathbb{E}[Y|G=g]$, we can define $X^g \in \{0,1\}$ to indicate whether or not an individual belongs to group $g$,

$$X^g = 1 \iff G = g$$

and then apply the bounds in Table 2 with respect to $X^g$ instead of $X$. Note that unlike the binary case, the relevant conditional associations will differ for each group. For example, the within-group association that determines the bias of the ecological regression estimator for $\mathbb{E}[Y|G=g]$ is given by

$$\delta_W^g = \mathbb{E}[\text{Cov}(Y, X_N^g | X^g)],$$

where $X_N^g = \mathbb{E}[X^g \mid N]$. Similarly, the between-group association that determines the bias of the neighborhood model estimator for $\mathbb{E}[Y|G=g]$ is given

by

$$\delta_B^g = \mathbb{E}[\mathrm{Cov}(Y, X^g|N)].$$

# C Proofs of Propositions

In this section, we prove the key results of the paper.

## Initial Results Relating to Ecological Regression and the Neighborhood Model

### Proposition 1

*Proof.* By the law of total covariance, we can decompose the covariance term in the numerator of $D_{ER}$ as:

$$\begin{aligned} D_{ER} &= \frac{\text{Cov}(Y, X_N)}{\text{Var}(X_N)} \\ &= \frac{\mathbb{E}[\text{Cov}(Y, X_N|X)] + \text{Cov}(\mathbb{E}[Y|X], \mathbb{E}[X_N|X])}{\text{Var}(X_N)} \\ &= \frac{\delta_W}{\text{Var}(X_N)} + \frac{\text{Cov}(\mathbb{E}[Y|X], \mathbb{E}[X_N|X])}{\text{Var}(X_N)} \end{aligned}$$

It remains to show that $\text{Cov}(\mathbb{E}[Y|X], \mathbb{E}[X_N|X]) = D \cdot \text{Var}(X_N)$. Let $\mu_x = \mathbb{E}[Y|X = x]$ and $\nu_x = \mathbb{E}[X_N|X = x]$. Since $X$ is binary, we can write:

$$\begin{aligned} \text{Cov}(\mathbb{E}[Y|X], \mathbb{E}[X_N|X]) &= \mathbb{E}[X](1 - \mathbb{E}[X])(\mu_1 - \mu_0)(\nu_1 - \nu_0) \\ &= \mathbb{E}[X](1 - \mathbb{E}[X]) \cdot D \cdot (\nu_1 - \nu_0) \end{aligned}$$

Using Bayes' Rule, we can write

$$\begin{aligned} \nu_x &= \mathbb{E}[X_N|X = x] \\ &= \sum_n X_n \, Pr\left(N = n \mid X = x\right) \\ &= \sum_n X_n \, \frac{Pr\left(X = x \mid N = n\right) \Pr[N = n]}{\Pr[X = x]} \end{aligned}$$

so that

$$\nu_1 = \frac{\mathbb{E}[X_N^2]}{\mathbb{E}[X]} \text{ and } \nu_0 = \frac{\mathbb{E}[X_N\left(1 - X_N\right)]}{1 - \mathbb{E}[X]}$$

This implies

$$\nu_1 - \nu_0 = \frac{\mathbb{E}[X_N^2] - \mathbb{E}[X]^2}{\mathbb{E}[X](1 - \mathbb{E}[X])}$$

Substituting this difference back into our covariance expression:

$$\begin{aligned}\text{Cov}(\mathbb{E}[Y|X], \mathbb{E}[X_N|X]) &= \mathbb{E}[X](1 - \mathbb{E}[X]) \cdot D \cdot \left(\frac{\mathbb{E}[X_N^2] - \mathbb{E}[X]^2}{\mathbb{E}[X](1 - \mathbb{E}[X])}\right) \\ &= D \cdot (\mathbb{E}[X_N^2] - \mathbb{E}[X]^2)\end{aligned}$$

Recognizing that $\mathbb{E}[X] = \mathbb{E}[X_N]$ by the law of iterated expectations, the term in parentheses is $\text{Var}(X_N)$. Thus:

$$\text{Cov}(\mathbb{E}[Y|X], \mathbb{E}[X_N|X]) = D \cdot \text{Var}(X_N)$$

Substituting this result into the expression for $D_{ER}$ yields $D_{ER} = D + \frac{\delta_W}{\text{Var}(X_N)}$. □

**Lemma 1**

*Proof.* We can write

$$\begin{aligned}D_{NM} &= \frac{\mathbb{E}[X_N Y_N]}{\mathbb{E}[X_N]} - \frac{\mathbb{E}[(1 - X_N)Y_N]}{\mathbb{E}[1 - X_N]} = \frac{\mathbb{E}[X_N Y_N] - \mathbb{E}[Y_N]\mathbb{E}[X_N]}{\mathbb{E}[X_N](1 - \mathbb{E}[X_N])} = \frac{\text{Cov}(X_N, Y_N)}{\mathbb{E}[X](1 - \mathbb{E}[X])} \\ &= \frac{\text{Cov}(X_N, Y_N)}{\text{Var}(X)}\end{aligned}$$

Multiplying and dividing by $\text{Var}(X_N)$ and noting that $D_{ER} = \frac{\text{Cov}(X_N, Y_N)}{\text{Var}(X_N)}$ yields the claim. □

**Proposition 2**

*Proof.* By Lemma 1 and the definition of $D_{ER}$, we can write

$$D_{NM} = \frac{\text{Cov}(X_N, Y_N)}{\text{Var}(X)}$$

Apply the law of iterated expectations to rewrite $\text{Cov}(X_N, Y_N)$ as $\text{Cov}(X, Y) - \mathbb{E}[\text{Cov}(X, Y|N)]$. Note that the latter term is $\delta_B$. Substitute this into $D_{NM}$, which gives

$$D_{NM} = \frac{\text{Cov}(X, Y) - \delta_B}{\text{Var}(X)}.$$

Recognizing $\frac{\text{Cov}(X,Y)}{\text{Var}(X)} = D$ and re-arranging yields the claim. □

**Proposition 3**

*Proof.* Substituting the expressions for $D_{ER}$ and $D_{NM}$ from Propositions 1 and 2 into Lemma 1 gives:

$$D - \frac{\delta_B}{\text{Var}(X)} = \gamma \left( D + \frac{\delta_W}{\text{Var}(X_N)} \right) \implies D(1-\gamma) = \frac{\delta_B}{\text{Var}(X)} + \frac{\gamma \delta_W}{\text{Var}(X_N)} = \frac{\delta_B + \delta_W}{\text{Var}(X)}.$$

□

## Results Relating to the Method of Bounds

The following results will be useful in subsequent proofs:

**Lemma C.1** (Feasibility of Intermediates)**.** *Suppose that for each $x$ and $n$, the conditional expectation of $Y$ with respect to $x$ and $n$ exists and is bounded, $\underline{Y} \leq \mathbb{E}[Y|X=x, N=n] \leq \overline{Y}$ for some pair $(\underline{Y}, \overline{Y}) \in \mathbb{R}^2$. Let $(A^1, A^0)$ and $(B^1, B^0)$ be two feasible pairs of group means, in the sense that there exist collections $\{A_n^1, A_n^0\}_{n \in \mathcal{N}}$ and $\{B_n^1, B_n^0\}_{n \in \mathcal{N}}$ satisfying, for each $n$,*

$$A_n^0, A_n^1, B_n^0, B_n^1 \in [\underline{Y}, \overline{Y}], \quad X_n A_n^1 + (1-X_n) A_n^0 = Y_n, \quad X_n B_n^1 + (1-X_n) B_n^0 = Y_n,$$

*and whose population average outcomes $(\mathbb{E}[Y|X=1], \mathbb{E}[Y|X=0])$ are given by $(A^1, A^0)$ and $(B^1, B^0)$ respectively.*

*Then:*

1. *For any $\alpha \in [0,1]$, the convex combination*

$$(Y^1, Y^0) = (\alpha A^1 + (1-\alpha) B^1,\ \alpha A^0 + (1-\alpha) B^0)$$

*is feasible, with associated disparity*

$$D^\alpha = \alpha D_A + (1-\alpha) D_B, \qquad D_A := A^1 - A^0,\ D_B := B^1 - B^0.$$

2. *If $D_A \leq D_B$ are two feasible values of the disparity, then every $D \in [D_A, D_B]$ is feasible. In particular, $D$ is attained by the mixture in part (i) with $\alpha = (D_B - D)/(D_B - D_A)$.*

*Proof.* (i) For any $\alpha \in [0,1]$, define neighborhood-level quantities

$$Y_n^{1,\alpha} := \alpha A_n^1 + (1-\alpha) B_n^1, \qquad Y_n^{0,\alpha} := \alpha A_n^0 + (1-\alpha) B_n^0.$$

Since $[\underline{Y}, \overline{Y}]$ is convex, $Y_n^{j,\alpha} \in [\underline{Y}, \overline{Y}]$ for $j \in \{0,1\}$. Moreover, by linearity,

$$X_n Y_n^{1,\alpha} + (1 - X_n) Y_n^{0,\alpha} = \alpha Y_n + (1-\alpha) Y_n = Y_n,$$

so the mixture is feasible. Aggregating over $n$ yields

$$Y^1 = \alpha A^1 + (1-\alpha) B^1, \qquad Y^0 = \alpha A^0 + (1-\alpha) B^0,$$

and hence $D^\alpha = Y^1 - Y^0 = \alpha D_A + (1-\alpha) D_B$.

(ii) If $D \in [D_A, D_B]$, then $\alpha = (D_B - D)/(D_B - D_A) \in [0,1]$. Part (i) implies that the corresponding mixture is feasible and has disparity

$$\alpha D_A + (1-\alpha) D_B = D.$$

□

We will refer to the constraint that $X_n Y_n^1 + (1 - X_n) Y_n^0 = Y_n$ as the "aggregation constraint".

**Lemma C.2** (Neighborhood Values Characterized by the Method of Bounds)**.** *Suppose that for each $x$ and $n$, the conditional expectation of $Y$ with respect to $x$ and $n$ exists and is bounded, $\underline{Y} \leq \mathbb{E}[Y|X = x, N = n] \leq \overline{Y}$ for some pair $(\underline{Y}, \overline{Y}) \in \mathbb{R}^2$. Fix a neighborhood $n \in \mathcal{N}$ with $\Pr(X = 1 \mid N = n) = X_n \in (0,1)$. Then the sharp bounds on the feasible values of $(Y_n^1, Y_n^0)$ are*

$$\begin{aligned}
Y_{\mathrm{MOB},n}^{1+} &:= \min\left\{ \frac{Y_n - \underline{Y}(1 - X_n)}{X_n},\ \overline{Y} \right\}, \\
Y_{\mathrm{MOB},n}^{0-} &:= \max\left\{ \frac{Y_n - \overline{Y} X_n}{1 - X_n},\ \underline{Y} \right\}, \\
Y_{\mathrm{MOB},n}^{1-} &:= \max\left\{ \frac{Y_n - \overline{Y}(1 - X_n)}{X_n},\ \underline{Y} \right\}, \\
Y_{\mathrm{MOB},n}^{0+} &:= \min\left\{ \frac{Y_n - \underline{Y} X_n}{1 - X_n},\ \overline{Y} \right\}.
\end{aligned}$$

*Proof.* Feasibility requires

$$Y_n = X_n Y_n^1 + (1 - X_n) Y_n^0, \qquad Y_n^1, Y_n^0 \in [\underline{Y}, \overline{Y}].$$

Solving the aggregation constraint for either variable gives

$$Y_n^1 = \frac{Y_n - (1 - X_n) Y_n^0}{X_n}, \qquad Y_n^0 = \frac{Y_n - X_n Y_n^1}{1 - X_n}.$$

*Maximizing disparity.* To maximize $Y_n^1 - Y_n^0$, assign as much of $Y_n$ as possible to group 1. The constraint $Y_n^0 \geq \underline{Y}$ implies

$$Y_n^1 \leq \frac{Y_n - \underline{Y}(1 - X_n)}{X_n},$$

while the boundedness constraint gives $Y_n^1 \leq \overline{Y}$. Thus,

$$Y_{\text{MOB},n}^{1+} = \min\left\{ \frac{Y_n - \underline{Y}(1 - X_n)}{X_n},\ \overline{Y} \right\}.$$

Substituting $Y_{MOB}^{1+}$ into the aggregation constraint solved for $Y_n^0$ yields:

$$Y_{\text{MOB},n}^{0-} = \frac{Y_n - X_n Y_{\text{MOB},n}^{1+}}{1 - X_n} = \max\left\{ \frac{Y_n - \overline{Y} X_n}{1 - X_n},\ \underline{Y} \right\}.$$

*Minimizing disparity.* By symmetry, minimizing $Y_n^1 - Y_n^0$ assigns as much of $Y_n$ as possible to group 0. The constraint $Y_n^1 \geq \underline{Y}$ implies

$$Y_n^0 \leq \frac{Y_n - \underline{Y} X_n}{1 - X_n},$$

together with $Y_n^0 \leq \overline{Y}$, yielding

$$Y_{\text{MOB},n}^{0+} = \min\left\{ \frac{Y_n - \underline{Y} X_n}{1 - X_n},\ \overline{Y} \right\}.$$

The corresponding value of $Y_n^1$ is

$$Y_{\text{MOB},n}^{1-} = \frac{Y_n - (1 - X_n) Y_{\text{MOB},n}^{0+}}{X_n} = \max\left\{ \frac{Y_n - \overline{Y}(1 - X_n)}{X_n},\ \underline{Y} \right\}.$$

By construction, the endpoint pairs are feasible, and every intermediate feasible pair is attainable as an $\alpha$-mixture of the endpoint constructions by Lemma C.1. These bounds are thus sharp.

□

**Proposition C.1** (Method of Bounds for Group Means)**.** *Suppose that for each $x$ and $n$, the conditional expectation of $Y$ with respect to $x$ and $n$ exists*

*and is bounded,* $\underline{Y} \leq \mathbb{E}[Y|X=x, N=n] \leq \overline{Y}$ *for some pair* $(\underline{Y}, \overline{Y}) \in \mathbb{R}^2$. *Define*

$$
\begin{aligned}
Y^{1+}_{\text{MOB}} &:= \frac{\mathbb{E}\big[\min\{\, Y_N - \underline{Y}(1-X_N),\, \overline{Y} X_N \,\}\big]}{\mathbb{E}[X]}, \\
Y^{0-}_{\text{MOB}} &:= \frac{\mathbb{E}\big[\max\{\, Y_n - \overline{Y} X_N,\, \underline{Y}(1-X_N) \,\}\big]}{1-\mathbb{E}[X]}, \\
Y^{1-}_{\text{MOB}} &:= \frac{\mathbb{E}\big[\max\{\, Y_N - \overline{Y}(1-X_N),\, \underline{Y} X_N \,\}\big]}{\mathbb{E}[X]}, \\
Y^{0+}_{\text{MOB}} &:= \frac{\mathbb{E}\big[\min\{\, Y_N - \underline{Y} X_N,\, \overline{Y}(1-X_N) \,\}\big]}{1-\mathbb{E}[X]}.
\end{aligned}
$$

*Then*

$$
Y^1 \in [\, Y^{1-}_{\text{MOB}},\, Y^{1+}_{\text{MOB}} \,], \tag{9}
$$

$$
Y^0 \in [\, Y^{0-}_{\text{MOB}},\, Y^{0+}_{\text{MOB}} \,]. \tag{10}
$$

*These bounds are sharp.*

*Proof.* Lemma C.2 characterizes, for each neighborhood $n$, the extreme feasible values of $(Y^1_n, Y^0_n)$ consistent with $(X_n, Y_n)$ and the outcome bounds.

*Derivation of the bounds.* Define $Y^{g\sigma}_{\text{MOB}} := \mathbb{E}[Y^{g\sigma}_{\text{MOB},N}]$ for $g \in \{0,1\}$ and $\sigma \in \{+,-\}$. We can write that:

$$
\mathbb{E}[Y^{g\sigma}_{MOB}] = \sum_{n \in \mathcal{N}} \Pr[N=n|X=g] Y^{g\sigma}_{MOB,n}
$$

Then noting that by Bayes' Rule:

$$
\Pr[N=n|X=g] = \frac{\Pr[X=g|N=n]\Pr[N=n]}{\Pr[X=g]}
$$

which is $X_n P_n / \mathbb{E}[X]$ for $X=1$ and $(1-X_n)P_n/(1-\mathbb{E}[X])$ for $X=0$. So we can write:

$$
\begin{aligned}
Y^{1+}_{MOB} &= \sum_{n \in \mathcal{N}} \frac{X_n P_n}{\mathbb{E}[X]} \cdot \min\left\{ \frac{Y_n - \underline{Y}(1-X_n)}{X_n}, \overline{Y} \right\} \\
&= \frac{1}{\mathbb{E}[X]} \sum_{n \in \mathcal{N}} P_n \min\left\{ Y_n - \underline{Y}(1-X_n), \overline{Y} X_n \right\} \\
&= \frac{\mathbb{E}\left[\min\left\{ Y_n - \underline{Y}(1-X_n), \overline{Y} X_n \right\}\right]}{\mathbb{E}[X]}.
\end{aligned}
$$

Similarly,

$$Y_{MOB}^{0-} = \sum_{n \in \mathcal{N}} \frac{P_n(1-X_n)}{1-\mathbb{E}[X]} \max\left\{\underline{Y}, \frac{Y_n - X_n\overline{Y}}{1-X_n}\right\}$$
$$= \frac{\mathbb{E}[\max\left\{\underline{Y}(1-X_n), Y_n - X_n\overline{Y}\right\}]}{1-\mathbb{E}[X]}$$

The remaining two expressions follow similarly.

*Sharpness.* Lemma C.2 implies that for each neighborhood $n$, the pairs $(Y_{\text{MOB},n}^{1+}, Y_{\text{MOB},n}^{0-})$ and $(Y_{\text{MOB},n}^{1-}, Y_{\text{MOB},n}^{0+})$ are feasible. Therefore the population-level pairs $(Y_{\text{MOB}}^{1+}, Y_{\text{MOB}}^{0-})$ and $(Y_{\text{MOB}}^{1-}, Y_{\text{MOB}}^{0+})$ are feasible as well.

For any $Y^1 \in [Y_{\text{MOB}}^{1-}, Y_{\text{MOB}}^{1+}]$, choose $\alpha \in [0,1]$ such that $Y^1 = \alpha Y_{\text{MOB}}^{1+} + (1-\alpha)Y_{\text{MOB}}^{1-}$. Applying Lemma C.1 neighborhood by neighborhood, the $\alpha$-mixture of the two endpoint constructions is feasible and achieves this value of $Y^1$ (with the corresponding $Y^0$). An identical argument applies to $Y^0$.

Hence the bounds in (9)–(10) are sharp. □

Note that setting $\underline{Y} = 0$,$\overline{Y} = 1$ yields the expressions for the MOB levels found in Section 5.

**Proposition C.2** (Method of Bounds for the Difference in Group Means)**.** *Suppose that for each $x$ and $n$, the conditional expectation of $Y$ with respect to $x$ and $n$ exists and is bounded, $\underline{Y} \leq \mathbb{E}[Y|X=x, N=n] \leq \overline{Y}$ for some pair $(\underline{Y}, \overline{Y}) \in \mathbb{R}^2$. Define*

$$D_{\text{MOB}}^{+} := \frac{\mathbb{E}\big[\min\{\, Y_n - \underline{Y}(1-X_n),\, \overline{Y}X_n \,\}\big] - \mathbb{E}[X]\mathbb{E}[Y]}{\text{Var}(X)},$$

$$D_{\text{MOB}}^{-} := \frac{\mathbb{E}[Y](1-\mathbb{E}[X]) - \mathbb{E}\big[\min\{\, Y_n - \underline{Y}X_n,\, \overline{Y}(1-X_n) \,\}\big]}{\text{Var}(X)}.$$

*Then:*

$$D \in [\, D_{\text{MOB}}^{-},\, D_{\text{MOB}}^{+} \,].$$

*These bounds are sharp.*

*Proof.* By construction, $D$ is maximized with $(Y_{MOB}^{1+}, Y_{MOB}^{0-})$ and minimized with $(Y_{MOB}^{1-}, Y_{MOB}^{0+})$. Consequently,

$$D \in \big[\, Y_{\text{MOB}}^{1-} - Y_{\text{MOB}}^{0+},\; Y_{\text{MOB}}^{1+} - Y_{\text{MOB}}^{0-} \,\big],$$

which is well-defined since by construction $Y^{1-}_{\text{MOB}} \le Y^{1+}_{\text{MOB}}$ and $Y^{0-}_{\text{MOB}} \le Y^{0+}_{\text{MOB}}$. Substituting the expressions for $Y^{1+}_{\text{MOB}}, Y^{1-}_{\text{MOB}}, Y^{0-}_{\text{MOB}}$, and $Y^{0+}_{\text{MOB}}$ from Proposition C.1 and simplifying yields the bounds stated in (C.2).

*Sharpness.* By Proposition C.1, the pairs

$$(Y^{1+}_{\text{MOB}}, Y^{0-}_{\text{MOB}}) \quad \text{and} \quad (Y^{1-}_{\text{MOB}}, Y^{0+}_{\text{MOB}})$$

are feasible. Hence the endpoint values $Y^{1+}_{\text{MOB}} - Y^{0-}_{\text{MOB}}$ and $Y^{1-}_{\text{MOB}} - Y^{0+}_{\text{MOB}}$ are attainable values of $D$.

Lemma C.1 then implies that every convex combination of these endpoints is also feasible. Therefore all values of $D$ in the interval

$$[\, Y^{1-}_{\text{MOB}} - Y^{0+}_{\text{MOB}},\ Y^{1+}_{\text{MOB}} - Y^{0-}_{\text{MOB}} \,]$$

are attainable, and the bounds are sharp. □

**Proposition 4**

*Proof.* The Proposition follows from setting $\underline{Y} = 0$, $\overline{Y} = 1$ in Proposition C.2. □

**Lemma C.3** (Width of MOB Interval)**.** *Suppose that for each $x$ and $n$, the conditional expectation of $Y$ with respect to $x$ and $n$ exists and is bounded, $\underline{Y} \le \mathbb{E}[Y|X = x, N = n] \le \overline{Y}$ for some pair $(\underline{Y}, \overline{Y}) \in \mathbb{R}^2$. Let $D^-_{\text{MOB}}$ and $D^+_{\text{MOB}}$ be the Method-of-Bounds (MOB) lower and upper bounds for $D = \mathbb{E}[Y \mid X = 1] - \mathbb{E}[Y \mid X = 0]$. Then*

$$D^+_{\text{MOB}} - D^-_{\text{MOB}} = \frac{1}{\text{Var}(X)}\,\mathbb{E}\Big[\min\Big\{(\overline{Y} - \underline{Y})X_N,\ (\overline{Y} - \underline{Y})(1 - X_N),\ Y_N - \underline{Y},\ \overline{Y} - Y_N\Big\}\Big].$$

*Proof.* Write $\Delta := \overline{Y} - \underline{Y} > 0$ and $a_N := Y_N - \underline{Y} \in [0, \Delta]$. Using the closed forms for the MOB bounds (from Proposition C.2), one can verify by direct algebra that

$$D^+_{\text{MOB}} - D^-_{\text{MOB}} = \frac{1}{\text{Var}(X)}\,\mathbb{E}\Big[m_1(N) + m_0(N) - Y_N\Big], \tag{11}$$

where

$$m_1(N) := \min\{\, Y_N - \underline{Y}(1 - X_N),\ \overline{Y}X_N \,\}, \qquad m_0(N) := \min\{\, Y_N - \underline{Y}X_N,\ \overline{Y}(1 - X_N) \,\}.$$

Now rewrite each term by pulling out the common $\underline{Y}$ parts:

$$\begin{aligned} m_1(N) &= \min\{\underline{Y}X_N + a_N,\ \underline{Y}X_N + \Delta X_N\} = \underline{Y}X_N + \min\{a_N, \Delta X_N\}, \\ m_0(N) &= \min\{\underline{Y}(1-X_N) + a_N,\ \underline{Y}(1-X_N) + \Delta(1-X_N)\} \\ &= \underline{Y}(1-X_N) + \min\{a_N, \Delta(1-X_N)\}. \end{aligned}$$

Therefore

$$m_1(N) + m_0(N) - Y_N = -\ a_N\ +\ \min\{a_N, \Delta X_N\}\ +\ \min\{a_N, \Delta(1-X_N)\}.$$

Define the normalized quantity $t_N := a_N/\Delta \in [0,1]$. Dividing by $\Delta$, the bracket equals

$$\Delta\Big(-t_N + \min\{t_N, X_N\} + \min\{t_N, 1-X_N\}\Big).$$

We claim that for any $x \in [0,1]$ and $t \in [0,1]$,

$$-t + \min\{t,x\} + \min\{t, 1-x\} = \min\{x, 1-x, t, 1-t\}. \tag{12}$$

To see this, let $m := \min\{x, 1-x\} \in [0, 1/2]$ and note that $\max\{x, 1-x\} = 1-m$. We consider three cases.

**Case 1:** $t \le m$**.** Then $t \le x$ and $t \le 1-x$, so $\min\{t,x\} = \min\{t, 1-x\} = t$ and the left-hand side equals $-t+t+t = t$. And $t \le m \le 1/2$ implies $t \le 1-t$. Hence the left-hand side=$\min\{x, 1-x, t, 1-t\}$. $t \le m \le 1/2$ also implies that $t \le 1-t$. In this case, the left-hand side is thus equal to $\min\{t, m, 1-t\}$.

**Case 2:** $m < t \le 1-m$**.** Then $\min\{t,x\} = m$ and $\min\{t, 1-x\} = t$ or vice versa (the smaller of $\{x, 1-x\}$ is $m$, while the larger is $1-m \ge t$). Hence the left-hand side equals $-t+m+t = m$. Note that $m < t \le 1$ implies $m < 1-t$, so the left-hand side equals $\min\{t, m, 1-t\}$.

**Case 3:** $t \ge 1-m$**.** Then $t \ge x$ and $t \ge 1-x$, so $\min\{t,x\} = x$ and $\min\{t, 1-x\} = 1-x$, and the left-hand side equals $-t+x+(1-x) = 1-t$. Note that $t \ge 1-m \ge 1/2$ implies $t \ge 1-t$. The left-hand side thus equals $\min\{t, m, 1-t\}$.

In the three cases, the left-hand side equals $\min\{t, m, 1-t\}$. Since $m = \min\{x, 1-x\}$, this is $\min\{x, 1-x, t, 1-t\}$, proving (12).

Applying (12) with $x = X_N$ and $t = t_N$ gives

$$\begin{aligned} m_1(N) + m_0(N) - Y_N &= \Delta \min\{X_N, 1 - X_N, t_N, 1 - t_N\} \\ &= \min\{\Delta X_N, \Delta(1 - X_N), a_N, \Delta - a_N\}. \end{aligned}$$

Since $a_N = Y_N - \underline{Y}$ and $\Delta - a_N = \overline{Y} - Y_N$, substituting back into (11) yields the stated formula. □

## Corollary 3

*Proof.* From Lemma 1 and the definition of $\gamma$, we can write

$$\begin{aligned} D_{NM} &= \frac{\mathrm{Cov}(X_N, Y_N)}{\mathrm{Var}(X)} = \frac{\mathbb{E}[X_N\, Y_N] - \mathbb{E}[X_N]\,\mathbb{E}[Y_N]}{\mathrm{Var}(X)} \\ &\leq \frac{\mathbb{E}[\min\{X_N,\, Y_N\}] - \mathbb{E}[X_N]\,\mathbb{E}[Y_N]}{\mathrm{Var}(X)} = D^{+}_{MOB} \end{aligned}$$

where the inequality follows from $X_n \leq 1$ and $Y_n \leq 1$ for all $n$. A similar argument establishes $D_{NM} \geq D^{-}_{MOB}$. □

## Corollary 4

*Proof.* The corollary follows from comparing $D_{ER}$ to the endpoints of the MOB interval and applying Proposition 1. □

### Lemma 2

*Proof.* The Lemma follows from setting $\underline{Y} = 0$, $\overline{Y} = 1$ in Lemma C.3. □

**Proposition C.3** (Characterizing the Width of Method of Bounds: General Case)**.** *Suppose that for each $x$ and $n$, the conditional expectation of $Y$ with respect to $x$ and $n$ exists and is bounded, $\underline{Y} \leq \mathbb{E}[Y|X = x, N = n] \leq \overline{Y}$ for some pair $(\underline{Y}, \overline{Y}) \in \mathbb{R}^2$. Assume $0 < \mathrm{Var}(X)$ and $\underline{Y} < \overline{Y}$.*

*Suppose in addition that $Y$ takes values only in $\{\underline{Y}, \overline{Y}\}$ (i.e. $Y$ is a two-point outcome on the endpoints). Then the width of the MOB identification interval for $D = \mathbb{E}[Y \mid X = 1] - \mathbb{E}[Y \mid X = 0]$ satisfies*

$$0 \leq D^{+}_{\mathrm{MOB}} - D^{-}_{\mathrm{MOB}} \leq 2 \min\left\{(\overline{Y} - \underline{Y})(1 - \gamma),\ \frac{\mathrm{Var}(Y)}{(\overline{Y} - \underline{Y})\,\mathrm{Var}(X)}\left(1 - \frac{\mathrm{Var}(Y_N)}{\mathrm{Var}(Y)}\right)\right\}.$$

*Proof.* Let $\Delta := \overline{Y} - \underline{Y} > 0$, and define the normalized outcome

$$\widetilde{Y} := \frac{Y - \underline{Y}}{\Delta} \in \{0, 1\},$$
$$\widetilde{Y}_N := \mathbb{E}[\widetilde{Y} \mid N] = \frac{Y_N - \underline{Y}}{\Delta}.$$

Then $\widetilde{D} := D/\Delta$, and the MOB width scales linearly:

$$D^+_{\text{MOB}} - D^-_{\text{MOB}} = \Delta\big(\widetilde{D}^+_{\text{MOB}} - \widetilde{D}^-_{\text{MOB}}\big).$$

By Lemma C.3 applied to $\widetilde{Y}$,

$$\widetilde{D}^+_{\text{MOB}} - \widetilde{D}^-_{\text{MOB}} = \frac{1}{\text{Var}(X)}\mathbb{E}\Big[\min\{X_N, 1 - X_N, \widetilde{Y}_N, 1 - \widetilde{Y}_N\}\Big].$$

Using $\min\{a, b, c, d\} \le \min\{a, b\}$ and $\min\{a, b, c, d\} \le \min\{c, d\}$ yields two upper bounds:

$$\widetilde{D}^+_{\text{MOB}} - \widetilde{D}^-_{\text{MOB}} \le \frac{1}{\text{Var}(X)}\mathbb{E}[\min\{X_N, 1 - X_N\}],$$
$$\widetilde{D}^+_{\text{MOB}} - \widetilde{D}^-_{\text{MOB}} \le \frac{1}{\text{Var}(X)}\mathbb{E}[\min\{\widetilde{Y}_N, 1 - \widetilde{Y}_N\}].$$

**First bound.** For any $u \in [0, 1]$, $\min\{u, 1 - u\} \le 2u(1 - u)$. Hence

$$\begin{aligned}\mathbb{E}[\min\{X_N, 1 - X_N\}] &\le 2\mathbb{E}[X_N(1 - X_N)] \\ &= 2\big(\text{Var}(X) - \text{Var}(X_N)\big) \\ &= 2\,\text{Var}(X)(1 - \gamma),\end{aligned}$$

which implies

$$\widetilde{D}^+_{\text{MOB}} - \widetilde{D}^-_{\text{MOB}} \le 2(1 - \gamma).$$

Multiplying by $\Delta$ gives

$$D^+_{\text{MOB}} - D^-_{\text{MOB}} \le 2\Delta(1 - \gamma).$$

**Second bound.** Since $\widetilde{Y} \in \{0, 1\}$,

$$\text{Var}(\widetilde{Y} \mid N) = \widetilde{Y}_N(1 - \widetilde{Y}_N).$$

Using again $\min\{u, 1-u\} \leq 2u(1-u)$,

$$\begin{aligned}\mathbb{E}[\min\{\widetilde{Y}_N, 1-\widetilde{Y}_N\}] &\leq 2\mathbb{E}[\widetilde{Y}_N(1-\widetilde{Y}_N)] \\ &= 2\mathbb{E}[\mathrm{Var}(\widetilde{Y} \mid N)] \\ &= 2\big(\mathrm{Var}(\widetilde{Y}) - \mathrm{Var}(\widetilde{Y}_N)\big),\end{aligned}$$

where the last equality follows from the law of total variance. Therefore,

$$\widetilde{D}^{+}_{\mathrm{MOB}} - \widetilde{D}^{-}_{\mathrm{MOB}} \leq \frac{2}{\mathrm{Var}(X)}\,\mathrm{Var}(\widetilde{Y})\left(1 - \frac{\mathrm{Var}(\widetilde{Y}_N)}{\mathrm{Var}(\widetilde{Y})}\right).$$

Rescaling back using $\mathrm{Var}(\widetilde{Y}) = \mathrm{Var}(Y)/\Delta^2$ and $\mathrm{Var}(\widetilde{Y}_N) = \mathrm{Var}(Y_N)/\Delta^2$, and multiplying by $\Delta$, yields

$$D^{+}_{\mathrm{MOB}} - D^{-}_{\mathrm{MOB}} \leq 2 \cdot \frac{\mathrm{Var}(Y)}{\Delta\,\mathrm{Var}(X)}\left(1 - \frac{\mathrm{Var}(Y_N)}{\mathrm{Var}(Y)}\right).$$

Taking the minimum of the two bounds completes the proof. □

**Proposition 5**

*Proof.* The Proposition follows from setting $\underline{Y} = 0$, $\overline{Y} = 1$ in Proposition C.3. □

## Results Relating to Bounds from Signs of Conditional Associations

### Theorem 1

Theorem 1 says that Table 1 provides sharp bounds on $D$; each row and column of the table correspond an assumption on the the sign (unknown, nonnegative,negative, zero) on $\delta_B$ and $\delta_W$, respectively. To prove this while keeping things manageable, we this into a series of claims which can be proven separately.

**Proposition C.4** (Bounds Based on the Sign of the Within-Group Association)**.**
*Suppose that $\gamma \in (0,1)$. Then: (i) If $\delta_W \geq 0$, then*

$$D \in \left[D^{-}_{MOB}\,,\; Min\left\{D^{+}_{MOB},\, D_{ER}\right\}\right]$$

*(ii) If $\delta_W \leq 0$, then*

$$D \in \left[ Max\left\{ D^-_{MOB},\, D_{ER} \right\} ,\, D^+_{MOB} \right]$$

*(iii) The bounds in (i) and (ii) are sharp.*

*Proof of Proposition C.4.* Recall that from Proposition 4, we have that $D \in [D^-_{MOB}, D^+_{MOB}]$, and Proposition 1 states that $D = D_{ER} - \frac{\delta_W}{\text{Var}(X_N)}$. Thus:

(i) If $\delta_W \geq 0$, then $D \leq D_{ER}$. Since $D \leq D^+_{MOB}$ and $D \leq D_{ER}$, $D \leq \min\{D_{ER}, D^+_{MOB}\}$. Combining this with $D \geq D^-_{MOB}$ yields the claimed bound.

(ii) If $\delta_W \leq 0$, then $D \geq D_{ER}$. Since $D \geq D^-_{MOB}$ and $D \geq D_{ER}$, $D \geq \max\{D_{ER}, D^-_{MOB}\}$. Combining this with $D \leq D^+_{MOB}$ yields the claimed bound

(iii) **Sharpness:** Consider case (i). By Proposition 4, any value $\tilde{D} \in [D^-_{MOB}, D^+_{MOB}]$ is achievable by some distribution $\tilde{Y}$ consistent with the observed data. For any such $\tilde{D}$ that also satisfies $\tilde{D} \leq D_{ER}$, the resulting within-group association $\delta_W = (D_{ER} - \tilde{D})\,\text{Var}(X_N) \geq 0$. Thus, every point in the interval stated in (i) is both feasible and consistent with the condition $\delta_W \geq 0$, making the bounds sharp. (ii) follows similarly.

□

**Proposition C.5** (Bounds Based on the Sign of the Between-Group Association)**.**
*Suppose that $\gamma \in (0, 1)$. Then: (i) If $\delta_B \geq 0$, then*

$$D \in \left[ D_{NM} \,,\, D^+_{MOB} \right]$$

*(ii) If $\delta_B \leq 0$, then*

$$D \in \left[ D^-_{MOB} \,,\, D_{NM} \right]$$

*(iii) The bounds in (i) and (ii) are sharp.*

*Proof of **Proposition C.5**.* Note that by definition of the method of bounds, $D \in [D^-_{MOB}, D^+_{MOB}]$ and by Corollary 3, $D_{NM} \in [D^-_{MOB}, D^+_{MOB}]$.

(i) Proposition 2 implies that if $\delta_B \geq 0$, $D \geq D_{NM}$. Since $D \geq D_{NM} \geq D^-_{MOB}$, (i) follows.

(ii) follows similarly: By Proposition 2, $\delta_B \leq 0 \implies D_{NM} \geq D$, and since $D \leq D_{NM} \leq D^+_{MOB}$, (ii) follows.

(iii) **Sharpness:** Any $\tilde{D}$ in the proposed intervals is achievable (Proposition 4). For any such $\tilde{D}$ in case (i), the implied association $\delta_B = (D_{NM} - \tilde{D})\operatorname{Var}(X_N)$ is non-negative by construction. Hence, each value in the interval in (i) is both feasible and satisfies $\delta_B \geq 0$. The proof of sharpness of case (ii) follows similarly.

□

*Proof of* ***Theorem 1****.* This Theorem is a consequence of the previous propositions, which show sharp bounds under the various possible conditions. First off, note that if $\delta_B, \delta_W \in \mathbb{R}$, then nothing is known besides what the method of bounds provide; thus Proposition 4 gives the top left cell. Now note that if $\delta_B = \delta_W = 0$ and $\gamma \neq 1$, then by Proposition 3, $D = 0$. If $\delta_W = 0$, $D = D_{ER}$, which takes care of the bottom row, while $\delta_B = 0 \implies D = D_{NM}$ takes care of the rightmost column. The cells with $\delta_{B \cdot W} \geq 0$ are given by Proposition 6. Proposition C.5 give the remaining two cells of the first row; Proposition C.4 gives the remaining two cells of the first column. For each of these cells, note that the we have already shown the bounds are sharp as in the respective propositions.

Two cells remain. For $\delta_B \geq 0$, $\delta_W \leq 0$, Proposition 1 shows that $D = D_{ER} - \frac{\delta_W}{\operatorname{Var}(X_N)} \geq D_{ER}$, while Proposition 2 shows that $D = D_{NM} + \frac{\delta_B}{\operatorname{Var}(X_N)} \geq D_{NM}$. Thus, $D \geq \max\{D_{ER}, D_{NM}\}$; but we also have that $D \leq D^+_{MOB}$, and $D \geq D^-_{MOB}$ by the method of bounds. Thus $D \geq \max\left\{D_{ER}, D_{NM}, D^-_{MOB}\right\}$ and $D \leq D^+_{MOB}$. Note that since the $D_{NM} \geq D^-_{MOB}$, we can drop $D^-_{MOB}$ from the maximum. Combining these yields the cell. To see that the bound is sharp, note first of all since the range is a subset of the method of bounds, Lemma C.1 shows that any $D$ does not conflict with $Y_n$ because it is a subset of the method of bounds range. Thus to be infeasible given the stated information, $D$ would have to result in $\delta_B < 0$ or $\delta_W > 0$; but neither of these can be true for the stated range (simply because we *derived* the stated range by identifying the region where these could not be true). The bound is thus sharp.

On the other hand, the same propositions show that $\delta_B \leq 0 \implies D \leq D_{NM}$, while $\delta_W \geq 0 \implies D \leq D_{ER}$. Again, we also have that $D^+_{MOB} \geq D$, so $\min\left\{D_{ER}, D_{NM}, D^+_{MOB}\right\} \geq D$, and again $D \geq D^-_{MOB}$. Again since

$D^+_{MOB} \geq D_{NM}$, we can drop $D^+_{MOB}$ from the minimum. Combining these yields the cell, sharpness follows similarly, and we have thus proved each cell in the table is as claimed. □

### Proposition 6

*Proof.* From Lemma 1, we know $D_{NM} = \gamma D_{ER}$ where $\gamma \in (0, 1)$. This implies $D_{NM}$ and $D_{ER}$ share a must share a sign and $|D_{NM}| < |D_{ER}|$, so it must be that either $0 \leq D_{NM} < D_{ER}$ or $D_{ER} < D_{NM} \leq 0$.

Now, by assumption $\delta_W \cdot \delta_B \geq 0$. That means either $\delta_W \geq 0$ and $\delta_B \geq 0$ or $\delta_W \leq 0$ and $\delta_B \leq 0$. If $\delta_W \geq 0$ and $\delta_B \geq 0$, then $D \geq D_{NM}$ by Proposition 2 and $D \leq D_{ER}$ by Proposition 1. Hence $D$ must lie between the two estimates, and we must have $D_{NM} \leq D \leq D_{ER}$, and this implies that $0 \leq D_{NM} \leq D \leq D_{ER}$. Since $D$ is also bounded by the MOB range $[D^-_{MOB}, D^+_{MOB}]$ and $D_{NM}$ is also always within the method of bounds, we can thus say that $0 \leq D_{NM} \leq D \leq \min\{D^+_{MOB}, E_{ER}\}$.

If instead, $\delta_W \leq 0$ and $\delta_B \leq 0$, then a similar argument, mutatis mutandis, shows the second part of the claim.

**Sharpness:** Any $D'$ in these intervals can be expressed as a convex combination of the endpoints. Since both endpoints are consistent with $\delta_W \cdot \delta_B \geq 0$, any point between them is also consistent with the sign constraints on the covariance terms. □

## Results Relating to Levels of Group Means

### Proposition 7

*Proof.* For Ecological Regression, note that we can rewrite:

$$Y^0_{ER} = \mathbb{E}[Y_N] - \left(\frac{\text{Cov}(Y_N, X_N)}{\text{Var}(X_N)}\right)\mathbb{E}[X_N] = \mathbb{E}[Y] - D_{ER}\mathbb{E}[X].$$

But then

$$Y^0_{ER} = \mathbb{E}[Y] - \left(D + \frac{\delta_W}{\text{Var}(X_N)}\right)\mathbb{E}[X]$$

using Proposition 1. Plugging in the fact that $\mathbb{E}[Y] = \mathbb{E}[X]Y^1 + (1 - \mathbb{E}[X])Y^0 = \mathbb{E}[X] \cdot D + Y^0$ yields:

$$\begin{aligned} Y^0_{ER} &= \mathbb{E}[X] \cdot D + Y^0 - D \cdot \mathbb{E}[X] - \frac{\delta_W}{\text{Var}(X_N)}\mathbb{E}[X] \\ &= Y^0 - \frac{\delta_W}{\text{Var}(X_N)}\mathbb{E}[X]. \end{aligned}$$

Re-arranging yields the claim for $Y^0_{ER} - Y^0$. A similar approach starting from $Y^1_{ER} = E[Y] + (1 - \mathbb{E}[X]) \cdot D_{ER}$ yields the claim for $Y_E R^1 - Y^1$.

For the neighborhood model: Notice that $Y^1 = \frac{\mathbb{E}[XY]}{\mathbb{E}[X]}$.Then:

$$Y^1_{NM} - Y^1 = \frac{\mathbb{E}[X_N Y_N]}{\mathbb{E}[X_N]} - \frac{\mathbb{E}[XY]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X_N Y_N] - \mathbb{E}[XY]}{\mathbb{E}[X]}$$

since $\mathbb{E}[X_N] = \mathbb{E}[X]$. Now we note that by the Law of Total Covariance, $\mathbb{E}[X_N Y_N] = \text{Cov}(X_N, Y_N) + \mathbb{E}[X_N]\mathbb{E}[Y_N] = \text{Cov}(X_N, Y_N) + \mathbb{E}[X]\mathbb{E}[Y]$, and substitute this in to obtain:

$$\frac{\text{Cov}(X_N, Y_N) + \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[XY]}{\mathbb{E}[X]} = \frac{\text{Cov}(X_N, Y_N) - \text{Cov}(X, Y)}{\mathbb{E}[X]}$$

But applying the Law of Total Covariance to $Cov(X, Y)$ shows that:

$$\text{Cov}(X, Y) = \mathbb{E}[\text{Cov}(X, Y|N)] + \text{Cov}(\mathbb{E}[X|N], \mathbb{E}[Y|N]) = \delta_B + \text{Cov}(X_N, Y_N);$$

substituting the right-hand side into the last equation yields the claim for $Y^1_{NM} - Y^1$. A similar approach yields $Y^0_{NM} - Y^0$. □

**Theorem 2**

*Proof.* The ideas in this proof are very similar to that of Theorem 1, so we will sketch out the key points without belaboring the details. We focus on Table 2; Table 3 follows similarly but with key signs reversed due to the difference in the biases given in Proposition 7.

If $\delta_W > 0$, then:

$$Y^1 = Y^1_{ER} - \frac{\delta_W(1 - \mathbb{E}[X])}{\text{Var}(X_N)} \leq Y^1_{ER}$$

If $\delta_B > 0$, then

$$Y^1 = Y^1_{NM} + \frac{\delta_B}{\mathbb{E}[X]} \geq Y^1_{NM}$$

Thus under CR with $\delta_W, \delta_B > 0$, we have that

$$Y^1_{NM} \leq Y^1_{NM} + \frac{\delta_B}{1 - \mathbb{E}[X]} \leq Y^1 \leq Y^1_{ER} - \frac{\delta_W(1 - \mathbb{E}[X])}{\text{Var}(X_N)} \leq Y^1_{ER}$$

Note that we must also have always $Y^1 \leq Y^{1+}_{MOB}$ and $Y^1 \geq Y^{1-}_{MOB}$ by the definition of the method of bounds; the neighborhood model is guaranteed to have $Y^1$ within the method of bounds estimates,, so we can omit those when the boundary is $Y^1_{NM}$, but we do need to compare it to $Y^1_{ER}$.

Similarly, if $\delta_W < 0$, then $Y^1 \geq Y^1_{ER}$, if $\delta_B < 0$ then $Y_1 \leq Y^1_{NM}$, and if both are true – i.e. contextual reinforcement in the negative direction – then $Y^1_{ER} \leq Y^1 \leq Y^1_{NM}$.

For the case in which $\delta_B \leq 0, \delta_W \geq 0$, note that in this case we will have $Y^1 \leq Y^1_{NM}$, $Y^1 \leq Y^1_{ER}$, and as always $Y^1 \leq Y^{1+}_{MOB}$, meaning it must be less than the minimum of the three; similarly with the maximum of $Y^1_{ER}$,$Y^1_{NM}$, and $Y^{1-}_{MOB}$ in the opposite case.

In cases where $\delta_W$ or $\delta_B$ are known to be exactly equal to zero, then the estimator coincides with the truth. Finally if both covariance terms are 0 and $\gamma \neq 1$, then by Theorem 1 we must have that $D = 0$, which implies $Y^1 = Y^0$.

And finally with regards to sharpness, we can use a similar argument as before to obtain any $Y^{1'}$ within the boundaries using a mixture; hence, the bounds are sharp.

□

## Results Relating to Neighborhood-Specific Identification

### Proposition 8

*Proof.* First recall that $D_n = \mu'(X_n) - \delta_{W,n}$, where $\mu'(X_n)$ is the derivative of the overall regression function. Now, suppose $\delta_{W,n} \geq 0$). This implies $D_n \leq \mu'(X_n)$. Intersecting this with the local Method of Bounds (MOB) interval $[D^-_{MOB,n}, D^+_{MOB,n}]$ yields:

$$D_n \in [D^-_{MOB,n}\,,\, \min\{D^+_{MOB,n}, \mu'(X_n)\}]$$

We determine the bounds for the group levels using the identities $Y_n^1 = Y_n + (1 - X_n)D_n$ and $Y_n^0 = Y_n - X_n D_n$:

- Since $D_n \leq \mu'(X_n)$, we have $Y_n^1 \leq Y_n + (1 - X_n)\mu'(X_n)$. Combined with the MOB upper bound, we get $Y_n^1 \leq \min\{Y_{MOB,n}^{1+}, Y_n + (1 - X_n)\mu'(X_n)\}$.
- Conversely, $D_n \leq \mu'(X_n)$ implies $-D_n \geq -\mu'(X_n)$, so $Y_n^0 \geq Y_n - X_n \cdot \mu'(X_n)$. Combined with the MOB lower bound, we get $Y_n^0 \geq \max\{Y_{MOB,n}^{0-}, Y_n - X_n \cdot \mu'(X_n)\}$.

On the other hand, if $\delta_{W,n} \leq 0$, then $D_n \geq \mu'(X_n)$. Reversing the inequalities in the logic above yields the mirrored bounds: $D_n \in [\max\{D_{MOB,n}^-, \mu'(X_n)\}, D_{MOB,n}^+]$, with corresponding shifts in the level bounds for $Y_n^1$ and $Y_n^0$.

**Sharpness:** Since the claimed intervals are subsets of the MOB range, any $D'$ in the range is consistent with the aggregate neighborhood data $Y_n$ and the global bounds $[\underline{Y}, \overline{Y}]$. To show sharpness, we note that the boundary where $\delta_{W,n} = 0$ corresponds to $D_n = \mu'(X_n)$. By Lemma C.1, since $\mu'(X_n)$ is a feasible disparity (by the property of the local linear approximation), any value between $\mu'(X_n)$ and the MOB endpoints is also feasible and maintains the assumed sign of $\delta_{W,n}$. Thus, the bounds are sharp. □

The following Proposition from [Manski, 2018] is restated and reproved for our framework:

**Proposition C.6** (Bounds Based on the Sign of the Neighborhood-Specific Between-Group Association [Manski, 2018] )**.**
*(i) If $\delta_{B,n} \geq 0$ for some $n \in \mathcal{N}$ then:*

$$\begin{aligned} D_n &\in [0, D_{MOB,n}^+], \\ Y_n^1 &\in [Y_n, Y_{MOB,n}^{1+}], \\ \text{and } Y_n^0 &\in [Y_{MOB,n}^{0-}, Y_n]. \end{aligned}$$

*(ii) If $\delta_{B,n} \leq 0$ for all $n \in \mathcal{N}$ then:*

$$\begin{aligned} D_n &\in [D_{MOB,n}^-, 0], \\ Y_n^1 &\in [Y_{MOB,n}^{1-}, Y_n], \\ \text{and } Y_n^0 &\in [Y_n, Y_{MOB,n}^{0+}]. \end{aligned}$$

*(iii) The bounds in (i) and (ii) are sharp.*

*Proof.* (i) Suppose that $\delta_{B,n} \geq 0$. Then $Y_n^1 \geq Y_n^0$, so $D_n \geq 0$. Combining this with the fact $D_n \leq D_{MOB,n}^+$ yields the stated interval. Now, to see that $Y_n^1 \geq Y_n$ and $Y_n^0 \leq Y_n$, note that $Y_n = X_n Y_n^1 + (1 - X_n)Y_n^0$ is a weighted average of $Y_n^1$ and $Y_n^0$. Thus, $Y_n^1 \geq Y_n^0$ implies that $Y_n^0 \leq Y_n \leq Y_n^1$. Combining these with the method of bounds yields the stated interval.

(ii) This follows similarly: $\delta_{B,n} \leq 0 \implies Y_n^1 \leq Y_n^0$, so $D_n \leq 0$ while also $D_n \geq D_{MOB,n}^-$. Again, $Y_n$ being a weighted average of $Y_n^1$ and $Y_n^0$ gives the remainder of the statement.

(iii) First note that $D_n$ is always within the method of bounds. Note that $D_n = 0 \iff \delta_{B,n} = 0$, so $D_n = 0$ is feasible. For the other end of the interval, $D_n = 0 \implies D_{MOB,n}^+ \geq 0$ (since $D_n = 0$ is feasible) and thus $D_{MOB,n}^+$ allows for $\delta_{B,n} \geq 0$ and is thus feasible as well. Lemma C.1 implies that all intermediate points are feasible as well.

Now consider $Y_n^1$. $Y_n^1 = Y_n$ is feasible because setting $Y_n^1 = Y_n = Y_n^0$ creates feasible disparity $D_n = 0$. We can also see that $Y_n^1 = Y_{MOB,n}^{1+}$ (paired with the complement $Y_n^0 = Y_{MOB,0}^-$) is also feasible, because it corresponds to $D_n = D_{MOB,n}^+ \geq 0$.

Since both $Y_n^1 = Y_n$ and $Y_n^1 = Y_{MOB,n}^{1+}$ are feasible; Lemma C.1 then gives that all points in the range are similarly feasible. The reasoning for the range containing $Y_n^0$ is symmetric. Finally, the argument for sharpness for (ii) is similar, mutatis mutandis.

□

**Proposition 9**

*Proof.* The bounds themselves follow by applying Propositions C.6 and 8 simultaneously. For sharpness we again focus on (i), as (ii) is similar. Suppose $\mu'(X_n) \geq 0$. First note that $D_n = 0$ is achieved by $Y_n^1 = Y_n^0 = Y_n \in [\underline{Y}, \overline{Y}]$ and since $\min\{\mu'(X_n), D_{MOB,n}^+\} \leq D_{MOB,n}^+$, which can be achieved by $Y_{MOB,n}^{1+}, Y_{MOB,n}^{0-}$, each $\tilde{D}_n$ in the interval be achieved by some distribution that is feasible in terms of $[\underline{Y}, \overline{Y}]$ and resulting in $Y_n$. So it is enough to show that any such distribution will result in $\text{Sign}(\delta_{B,n}) = \text{Sign}(\delta_{W,n})$. But this holds because $\delta_{W,n} \geq 0 \iff D_n \leq \mu'(X_n)$, and $\delta_{B,n} \geq 0 \iff D_n \geq 0$, and both hold on the interior of the interval. A similar argument can be made for the bounds on $Y_n^1$ and $Y_n^0$.

□

# D Supplemental Figures and Tables

| Tier | Uniquely Matched | Unmatched | Proportion Matched |
|---|---|---|---|
| Tier 1 | 1,028,172 | 764,810 | .569 |
| Tier 2 | 61,906 | 712,439 | .603 |
| Tier 3 | 129,298 | 582,697 | .675 |
| Tier 4 | 37,409 | 545,694 | .695 |
| Tier 5 | 35,231 | 510,427 | .715 |
| Tier 6 | 89,605 | 420,430 | .764 |

Table S1: Summary of matched patient counts for the auxiliary individual-level dataset. Tier 1 contains the highest confidence matches and Tier 6 contains the lowest confidence matches (for details, see Appendix E). For individuals with multiple matches, one match is chosen at random. The proportion is the proportion of patients matched from the vaccine data.

| Tier | Uniquely Matched | Unmatched | Proportion Matched |
|---|---|---|---|
| Tier 1 | 1,017,300 | 764,810 | 0.563 |
| Tier 2 | 57,517 | 702,882 | 0.595 |
| Tier 3 | 101,999 | 573,883 | 0.651 |
| Tier 4 | 39,042 | 563,416 | 0.673 |
| Tier 5 | 26,207 | 511,227 | 0.687 |
| Tier 6 | 39,464 | 414,351 | 0.709 |

Table S2: Summary of matched patient counts for the auxiliary individual-level dataset. Tier 1 contains the highest confidence matches and Tier 6 contains the lowest confidence matches. Individuals with multiple matches are discarded. The proportion is the proportion of patients matched from the vaccine data.

| Discard ties<br>Equal weight | **Tier 1** | **Tier 1-2** | **Tier 1-3** | **Tier 1-4** | **Tier 1-5** | **Tier 1-6** |
|---|---|---|---|---|---|---|
| $E[Cov(Y, X_N \mid X)]$ | -0.00463<br>(8e-05) | -0.00452<br>(8e-05) | -0.00451<br>(7e-05) | -0.00447<br>(7e-05) | -0.00455<br>(7e-05) | -0.00466<br>(7e-05) |

| | | | | | | |
|---|---|---|---|---|---|---|
| $E[Cov(Y, X\|N)]$ | -0.00691<br>(0.00022) | -0.00703<br>(0.00021) | -0.0066<br>(0.0002) | -0.00652<br>(0.0002) | -0.00623<br>(0.0002) | -0.00592<br>(0.0002) |

Table S3: Conditional covariance (standard error) estimated from the auxiliary individual-level dataset. $Y$ indicates whether an individual is vaccinated, $X$ indicates whether they are Republican, and $X_N$ is the proportion of Republicans in their county. Individuals all have equal weight and are matched in a tiered approach, where Tier 1 contains the highest confidence matches and Tier 6 contains the lowest confidence matches. Individuals with multiple matches are discarded.

| Random tiebreak<br>Equal weight | **Tier 1** | **Tier 1-2** | **Tier 1-3** | **Tier 1-4** | **Tier 1-5** | **Tier 1-6** |
|---|---|---|---|---|---|---|
| $E[Cov(Y, X_N\|X)]$ | -0.00463<br>(8e-05) | -0.00451<br>(8e-05) | -0.0045<br>(7e-05) | -0.00446<br>(7e-05) | -0.00459<br>(7e-05) | -0.00481<br>(7e-05) |
| $E[Cov(Y, X\|N)]$ | -0.00698<br>(0.00022) | -0.00706<br>(0.00021) | -0.00638<br>(0.0002) | -0.00633<br>(0.0002) | -0.00594<br>(0.00019) | -0.00519<br>(0.00019) |

Table S4: Conditional covariance (standard error) estimated from the auxiliary individual-level dataset. $Y$ indicates whether an individual is vaccinated, $X$ indicates whether they are Republican, and $X_N$ is the proportion of Republicans in their county. Individuals all have equal weight and are matched in a tiered approach, where Tier 1 contains the highest confidence matches and Tier 6 contains the lowest confidence matches. For individuals with multiple matches, one match is chosen at random.

| Discard ties<br>Re-weighted | **Tier 1** | **Tier 1-2** | **Tier 1-3** | **Tier 1-4** | **Tier 1-5** | **Tier 1-6** |
|---|---|---|---|---|---|---|
| $E[Cov(Y, X_N\|X)]$ | -0.00361<br>(0.00042) | -0.00363<br>(0.00041) | -0.00353<br>(0.00038) | -0.00354<br>(0.00037) | -0.0036<br>(0.00037) | -0.00366<br>(0.00036) |

| | | | | | | |
|---|---|---|---|---|---|---|
| $E[Cov(Y, X\|N)]$ | -0.00847<br>(0.001) | -0.00876<br>(0.00096) | -0.00833<br>(0.00089) | -0.00712<br>(0.0009) | -0.00691<br>(0.00088) | -0.00656<br>(0.00087) |

Table S5: Conditional covariance (standard error) estimated from the auxiliary individual-level dataset. $Y$ indicates whether an individual is vaccinated, $X$ indicates whether they are Republican, and $X_N$ is the proportion of Republicans in their county. Individuals are re-weighted (see Appendix E) and are matched in a tiered approach, where Tier 1 contains the highest confidence matches and Tier 6 contains the lowest confidence matches. Individuals with multiple matches are discarded.

# E Auxiliary Matched Analysis: Additional Detail

## Data

### Individual COVID-19 Vaccinations

We used data on individual COVID-19 vaccination status from the American Family Cohort (AFC). The AFC is derived from the American Board of Family Medicine PRIME Registry [Balraj et al., 2023]. PRIME is used by practices to track electronic health records data for Medicare and Medicaid reporting as well as measurement of social determinants of health [Phillips, 2017]. PRIME contains approximately 8 million patient records across 47 states [Balraj et al., 2023]. From this dataset, we obtained COVID-19 vaccination status following Hao et al., 2023 [Hao et al., 2023]. We considered all patients over 5 years old who did not die during 2021. Patients who had no history of receiving immunizations or visiting practices operating from January 1, 2019 to December 31, 2021 were excluded. We considered patients to be vaccinated if they received two COVID-19 vaccinations, or a single dose of the Janssen vaccine, before December 18, 2021 (n = 339,484). All other patients (n = 1,463,303) were considered unvaccinated.

### Individual Voter Information

We used national voter records from the L2 database [L2, 2024]. This database is collected and assembled by the L2 organization and sold for use in academic, political, or business settings. In addition to minimizing data gathering steps, the database was constructed using a number of standardization and cleaning preprocessing steps that ensure greater comparability across states. We obtained voter name, state, date of birth, and registered political party for 214,125,844 voters nationally.

### County-Level Voter Information

We used election returns at the county level from the 2020 Presidential Election to provide the $b_i$ in the individual-level analysis to check the signs of the covariance terms [MIT Election Data and Science Lab, 2020].

### County-Level COVID-19 Vaccination Information

We measured county-level vaccination uptake as of December 31, 2021 using data from the Centers for Disease Control and Prevention [Centers for Disease Control and Prevention, 2024].

## Matching

We used a tiered matching strategy to pair patients from the AFC dataset with voters from the L2 dataset. This strategy was informed by [Roos Jr et al., 1986], who implement a tiered matching system for records with few identifiers. We implement a similar system, where "Tier 1" matches represent highest confidence, and "Tier 6" matches represent lowest confidence (see Table S2, S1).

We used the characteristics of the datasets to determine which factors to match on in each tier. In the L2 data, we found that recorded birth dates were non-uniformly distributed, with strong biases towards birth dates on the first of each month and on January 1. This suggests that the L2 data may record births as occurring on the first of the month or on January first when respondents only note their birth month or year. Thus, in some tiers, we allow for matches without identical birth dates as long as the birth month and year are the same.

We require an exact match on first and last name in all tiers. We considered matching on first initial and last name, but we found that these matching schemes resulted in clearly erroneous matches. We also considered matching on phonetic last name. However, since our datasets come from official voter and patient records, it is unlikely that last names are misspelled. Thus, a phonetic encoding may have led to incorrect matches.

In our final matching scheme, tier 1 requires matches at the highest level of specificity with exact matches across all of our chosen factors. We are most confident that tier 1 matches are correct, since they match for all six factors. Tier 6 only requires exact matches on three factors, so we are less confident that these matches are correct. We note, in particular, that our Covid-19 vaccination data contain data from individuals in 47 states, and our matched dataset ultimately contains individuals from 49 states, where individuals matched to these two additional states must have been from the lower confidence tier matches (Tiers 4, 5, and 6), which do not require exact matching on state. We believe this choice is justified to maximize the size of the matched dataset. The tiers are defined as follows:

- Tier 1: Unique match on exact first name, last name, state, and birth date (day/month/year)
- Tier 2: Unique match on exact first name, last name, state, and birth month/year
- Tier 3: Unique match on exact first name, last name, state, and birth year
- Tier 4: Unique match on exact first name, last name, and birth date (day/month/year)
- Tier 5: Unique match on exact first name, last name, and birth month/year
- Tier 6: Unique match on exact first name, last name, and birth year

We tested two different ways of considering ties, where multiple vaccination records or voters were matched to one voter or vaccination record in a given tier. In the first approach, we did not allow any ties ("zero-tie"), deleting ties from matched data at each tier (Tables S3, S5). In the second approach, following [Roos Jr et al., 1986], we kept one record at random from each set of ties ("random-tie") (Table S4). We present analyses of the

zero-tie strategy in the main text. Analyses for the random-tie strategy are presented in the supplement.

When estimating covariances and confidence intervals, we tested two different weighting schemes. In the first scheme, each record was given equal weight ("unit weight"). In the second scheme, we re-weighted records such that the sum of records for a given county in a given political party matched the number of voters of that party in that county, as calculated from our county-level voter information ("re-weighted"). Because many counties had few records in our matched dataset, we restricted the re-weighted data to only weight patients coming from counties with over 20 records – all other patients were given weight 0. We present analyses for the unit weight strategy in the main text. Analyses for the re-weighted strategy are presented in the supplement.

When patient counties recorded in the AFC and L2 data did not match, we chose to use the AFC county, since the AFC county represents a more recent county affiliation than the L2 county. For the zero-tie and random-tie approaches, 4,346 and 5,235 records respectively were removed due to missing AFC county. Additionally, in Alaska, the county-level dataset recorded Alaskan districts rather than counties. We removed all people with AFC counties in Alaska from the merged dataset. We also restricted to only Democrat and Republican voters. This left us with a total of 987,663 (437,722 Democrat, 549,941 Republican) and 1,061,517 (476,496 Democrat, 585,021 Republican) individuals matched in the zero-tie and random-tie approaches, respectively.

# F Estimating Conditional Covariances

In this section, we describe how we estimate $\theta = \mathbb{E}[\text{Cov}(A, B|C)]$, where $C$ takes values in $\mathcal{C}$, and $\text{Var}(\hat{\theta})$. We assume that the distribution of $C$ is known, as is the distribution of $A|C$.

We know:

$$\theta = \sum_{c \in \mathcal{C}} \Pr(C = c) \, \text{Cov}(A, B|C = c)$$

which we estimate by

$$\begin{aligned}
\hat{\theta} &= \sum_{c \in \mathcal{C}} \Pr(C = c) \sum_{i:C_i = c} \frac{1}{n_c - 1}(A_i - \bar{A})(B_i - \bar{B}) \\
&= \sum_{c \in \mathcal{C}} \frac{\Pr(C = c)}{n_c - 1} \sum_{i:C_i = c} (A_i - \bar{A})(B_i - \bar{B}) \\
&= \sum_{c \in \mathcal{C}} \Pr(C = c) \left( \frac{\sum_{i:C_i = c} (A_i - \bar{A})^2}{n_c - 1} \right) \left( \frac{\sum_{i:C_i = c} (A_i - \bar{A})(B_i - \bar{B})}{\sum_{i:C_i = c} (A_i - \bar{A})^2} \right) \\
&= \sum_{c \in \mathcal{C}} \Pr(C = c) \, \text{Var}(A|C = c) \hat{\beta}_c
\end{aligned}$$

where $\hat{\beta}_c$ is the coefficient from the linear regression of $A$ on $B$ when $C = c$, $n_c$ is the number of entries where $C = c$, and $\text{Var}(A|C = c)$ is the variance of $A$ for entries where $C = c$.

We now estimate $\text{Var}(\hat{\theta})$.

$$\begin{aligned}
\text{Var}(\hat{\theta}) &= \text{Var}(\sum_{c \in \mathcal{C}} \Pr(C = c) \, \text{Var}(A|C = c) \hat{\beta}_c) \\
&= \sum_{c \in \mathcal{C}} \text{Var}(\Pr(C = c) \, \text{Var}(A|C = c) \hat{\beta}_c) \\
&= \sum_{c \in \mathcal{C}} \Pr(C = c)^2 \, \text{Var}(A|C = c)^2 \, \text{Var}(\hat{\beta}_c) \\
&= \sum_{c \in \mathcal{C}} \Pr(C = c)^2 \, \text{Var}(A|C = c)^2 \text{SE}(\hat{\beta}_c)^2
\end{aligned}$$

Where $\text{SE}(\beta_c)^2$ is the estimated standard error from the linear regression of $A$ on $B$ when $C = c$.

In our setting, we seek to estimate $\mathbb{E}\left[\text{Cov}(X_N, Y|X)\right]$ and $\mathbb{E}\left[\text{Cov}(X, Y|X_N)\right]$. We estimate $\mathbb{E}\left[\text{Cov}(X_N, Y|X)\right]$ as follows:

We calculate $Pr(X = 1)$ as the weighted sum of entries where $X = 1$ over the total weight of the dataset. We then calculate $Var(X_N|X = i)$. For each value of $X$, we then perform weighted least squares regression ($Y \sim X_N|X = i$). We calculate the coefficient on $X_N$ of this regression, $\hat{\beta}_i$. For our estimate for the standard error of $\mathbb{E}\left[\text{Cov}(X_N, Y|X)\right]$, we calculate the standard error of $\hat{\beta}_i$ and follow the formulas above.

For $\mathbb{E}\left[\text{Cov}(X, Y|X_N)\right]$, we first note a useful proposition:

**Proposition F.1.** *We can write that:*

$$\mathbb{E}[\text{Cov}(X, Y|N)] = \mathbb{E}[\text{Cov}(X, Y|X_N)]$$

Using this proposition, we estimate $\mathbb{E}\left[\text{Cov}(X, Y|N)\right] = \mathbb{E}\left[\text{Cov}(X, Y|X_N)\right]$ as follows:

We round $X_N$ to the nearest .05, creating 19 groups $(d_1, ...d_{19})$ of $X_N$ values (in this case, $X_N$ = county proportion Republican). We estimate $Pr(X_N \in d_i)$ as the weighted sum of entries with $X_N \in d_i$ over the total weight of the dataset. We also calculate the variance of $X$, $\text{Var}(X|X_N \in d_i)$. We then perform weighted least squares regression ($Y \sim X|X_N \in d_i$) and calculate the coefficient on $X$ of this regression, $\widehat{\beta_{d_i}}$. For our estimate for the standard error of $\mathbb{E}\left[\text{Cov}(X, Y|X_N)\right]$, we calculate the standard error of $\widehat{\beta_{d_i}}$, $\text{SE}(\widehat{\beta_{d_i}})$ and follow the formulas above.

*Proof of Proposition F.1.* By the Law of Total Covariance, the equality $\mathbb{E}[\text{Cov}(X, Y|N)]$ $= \mathbb{E}[\text{Cov}(X, Y|X_N)]$ holds if and only if the between-group covariances are equal:

$$\text{Cov}(\mathbb{E}[X|N], \mathbb{E}[Y|N]) = \text{Cov}(\mathbb{E}[X|X_N], \mathbb{E}[Y|X_N]) \tag{13}$$

By the definition of the neighborhood mean composition, $\mathbb{E}[X|N] = X_N$. Furthermore, since $X_N$ is a function of $N$, the tower property implies $\mathbb{E}[X|X_N] = \mathbb{E}[\mathbb{E}[X|N]|X_N] = X_N$. Thus, Equation (13) simplifies to:

$$\text{Cov}(X_N, Y_N) = \text{Cov}(X_N, \mathbb{E}[Y|X_N])$$

Expanding the covariance, it is sufficient to show that $\mathbb{E}[X_N Y_N] = \mathbb{E}[X_N \mathbb{E}[Y|X_N]]$. Applying the Law of Iterated Expectations to the left-hand side:

$$\begin{aligned}
\mathbb{E}[X_N Y_N] &= \mathbb{E}[\mathbb{E}[X_N Y_N \mid X_N]] \\
&= \mathbb{E}[X_N \mathbb{E}[Y_N \mid X_N]] && (X_N \text{ is constant given } X_N) \\
&= \mathbb{E}[X_N \mathbb{E}[\mathbb{E}[Y|N] \mid X_N]] && (\text{Definition of } Y_N) \\
&= \mathbb{E}[X_N \mathbb{E}[Y \mid X_N]] && (\text{Tower Property})
\end{aligned}$$

where the final equality follows because $\sigma(X_N) \subseteq \sigma(N)$. This confirms the equality of the expectations and completes the proof. □

# G Relaxing Assumptions to Recover Neighborhood - Specific Groups

Now we relax the assumption we made in Section 6 that any two neighborhoods that have the same group prevalence will also share the same expected outcome by group; relaxing this means allowing that neighborhoods with the same group prevalence have different expected outcomes. In this case, the analogues to Propositions C.6 and 9 apply in an average sense; i.e. we can obtain upper and lower bounds on the average disparity and average values of $Y_n^j$ over the set of neighborhoods with the same group prevalence.

For each of our random variables, we use the notation $\tilde{\cdot}_n$ to indicate that we are averaging over the set of neighborhoods $n' \in \mathcal{N}$ such that $X_{n'} = X_n$. For example:

$$\begin{aligned}\tilde{Y}_n^1 :&= \mathbb{E}[Y_n^1 | n \in \mathcal{N}^{-1}(X_n)] \\ &= \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n^1\end{aligned}$$

where $\mathcal{N}^{-1}(x_n)$ is the set of $n \in \mathcal{N}$ such that $X_n = x_n$ and $w_n = p_n / \sum_{n \in \mathcal{N}^{-1}(x_n)} p_n$. We define $\tilde{Y}_n^0$ similarly, and $\tilde{D}_n$ as $\tilde{Y}_n^1 - \tilde{Y}_n^0$. We can also define the method of bounds similarly — $\tilde{Y}_{MOB,n}^{1+} = \sum_{n \in \mathcal{N}^{1+}(x_n)} w_n Y_{MOB,n}^{1+}$ and so on, and immediately have that:

$$\begin{aligned}\tilde{Y}_n^1 &\in [\tilde{Y}_{MOB,n}^{1-}, \tilde{Y}_{MOB,n}^{1+}], \\ \tilde{Y}_n^0 &\in [\tilde{Y}_{MOB,n}^{0-}, \tilde{Y}_{MOB,n}^{0+}],\end{aligned}$$

because

$$Y_n^1 \leq Y_{MOB,n}^{1+} \; \forall n \in \mathcal{N}^{-1}(x_n) \implies \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n^1 \leq \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_{MOB,n}^{1+} = \tilde{Y}_{MOB,n}^{1+}$$

and so on; similarly, we can write that:

$$\begin{aligned}\tilde{D}_n \in &[\tilde{D}_{MOB,n}^-, \tilde{D}_{MOB,n}^+] \\ =&[\tilde{Y}_n^{1-} - \tilde{Y}_n^{0+}, \tilde{Y}_n^{1+} - \tilde{Y}_n^{0-}].\end{aligned}$$

These define the method of bounds. We now turn to the equivalents of Propositions C.6-9. As before, define $\mu(x_n) := \mathbb{E}[Y | X_n = x_n]$. Define also

$$\tilde{\delta}_{B,n} := \text{Var}[X | N = n] \left( \tilde{Y}_n^1 - \tilde{Y}_n^0 \right)$$

and

$$\tilde{\delta}_{W,n} := \sum_{n \in \mathcal{N}^{-1}(X_n)} w_n \tilde{\delta}_{W,n}.$$

Note first:

**Lemma G.1.** *Define $\tilde{\delta}_{W,n}$, $\tilde{D}_n$ as above. Then:*

$$\tilde{D}_n = \mu'(X_n) - \tilde{\delta}_{W,n}$$

*Proof.* Note that:

$$\mu(x_n) = \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n;$$

this can be re-written as:

$$\begin{aligned}\mu(x_n) &= \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n \cdot (X_n Y_n^1 + (1 - X_n) Y_n^0) \\ &= X_n \cdot \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n^1 + (1 - X_n) \cdot \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n^0\end{aligned}$$

since $X_n$ is constant for all $n \in \mathcal{N}^{-1}(x_n)$. Then taking the derivative gives:

$$\begin{aligned}\mu'(X_n) &= X_n \cdot \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n^{1'} + (1 - X_n) \cdot \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n Y_n^{0'} + \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n (Y_n^1 - Y_n^0) \\ &= \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n \left(X_n \mu_1'(x_n) + (1 - X_n) \mu_0'(x_n)\right) + \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n (Y_n^1 - Y_n^0) \\ &= \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n \delta_{W,n} + \sum_{n \in \mathcal{N}^{-1}(x_n)} w_n (Y_n^1 - Y_n^0) \\ &= \tilde{\delta}_{W,n} + \tilde{D}_n.\end{aligned}$$

□

We can now state the generalized versions of Propositions C.6-9:

**Proposition G.1** (General version of Proposition C.6)**.** *(i) If* $\tilde{\delta}_{B,n} \geq 0$ *for some set of neighborhoods* $\mathcal{N}^{-1}(x_n)$*, then:*

$$\begin{aligned}\tilde{D}_n &\in [0, \tilde{D}^{+}_{MOB,n}] \\ \tilde{Y}^1_n &\in [\tilde{Y}_n, \tilde{Y}^{1+}_{MOB,n}] \\ \tilde{Y}^0_n &\in [\tilde{Y}^{0-}_{MOB,n}, \tilde{Y}_n]\end{aligned}$$

*(ii) If* $\tilde{\delta}_{B,n} \leq 0$ *for some set of neighborhoods* $\mathcal{N}^{-1}(x_n)$*, then:*

$$\begin{aligned}\tilde{D}_n &\in [\tilde{D}^{-}_{MOB,n}, 0], \\ \tilde{Y}^1_n &\in [\tilde{Y}^{1-}_{MOB,n}, \tilde{Y}_n] \\ \textit{and } \tilde{Y}^0_n &\in [\tilde{Y}_n, \tilde{Y}^{0+}_{MOB,n}].\end{aligned}$$

*(iii) The bounds in (i) and (ii) are sharp in the absence of additional information.*

**Proposition G.2** (General version of Proposition 8)**.** *(i)Suppose that* $\tilde{\delta}_{W,n} \geq 0$ *for some neighborhoods* $\mathcal{N}^{-1}(x_n)$*. Then:*

$$\begin{aligned}\tilde{Y}_n &\in \left[\tilde{D}^{-}_{MOB,n}, \min\left\{\mu'(X_n), \tilde{D}^{+}_{MOB,n}\right\}\right], \\ Y^1_n &\in \left[\tilde{Y}^{1-}_{MOB,n}, \min\left\{\tilde{Y}^{1+}_{MOB,n}, \tilde{Y}_n + (1 - X_n)\mu'(X_n)\right\}\right], \\ \textit{and } \tilde{Y}^0_n &\in \left[\max\left\{\tilde{Y}^{0-}_{MOB,n}, \tilde{Y}_n - X_n \cdot \mu'(X_n)\right\}\right]\end{aligned}$$

*(ii) Suppose that* $\tilde{\delta}_{W,n} \leq 0$ *for some set of neighborhoods* $\mathcal{N}^{-1}(x_n)$*. Then:*

$$\begin{aligned}\tilde{Y}_n &\in \left[\max\left\{\mu'(x_n), \tilde{D}^{-}_{MOB,n}\right\}, \tilde{D}^{+}_{MOB,n}\right] \\ \tilde{Y}^1_n &\in \left[\max\left\{\tilde{Y}^{1-}_{MOB,n}, \tilde{Y}_n + (1 - X_n) \cdot \mu'(X_n)\right\}, \tilde{Y}^{1+}_{MOB,n}\right] \\ Y^0_n &\in \left[\tilde{Y}^{0-}_{MOB,n}, \min\left\{\tilde{Y}^{0+}_{MOB,n}, \tilde{Y}_n - X_n \cdot \mu'(X_n)\right\}\right]\end{aligned}$$

*(iii) The bounds in (i) and (ii) are sharp in the absence of further information.*

**Proposition G.3** (General Version of Proposition 9)**.** *Suppose that* $\tilde{\delta}_{B,n} \cdot \tilde{\delta}_{W,n} \geq 0$*. Then either:*

*(i)* $\mu'(X_n) \geq 0$, *and:*

$$\begin{aligned}
\tilde{D}_n &\in [0, \min\{\mu'(X_n), \tilde{D}^+_{MOB,n}] \\
Y^1_n &\in \left[\tilde{Y}_n, \min\{\tilde{Y}_n + (1 - X_n) \cdot \mu'(X_n), \tilde{Y}^{1+}_{MOB,n}\right] \\
\tilde{Y}^0_n &\in \left[\tilde{Y}_n, \min\{\tilde{Y}_n - X_n \mu'(X_n), \tilde{Y}^{0+}_{MOB,n}\right];
\end{aligned}$$

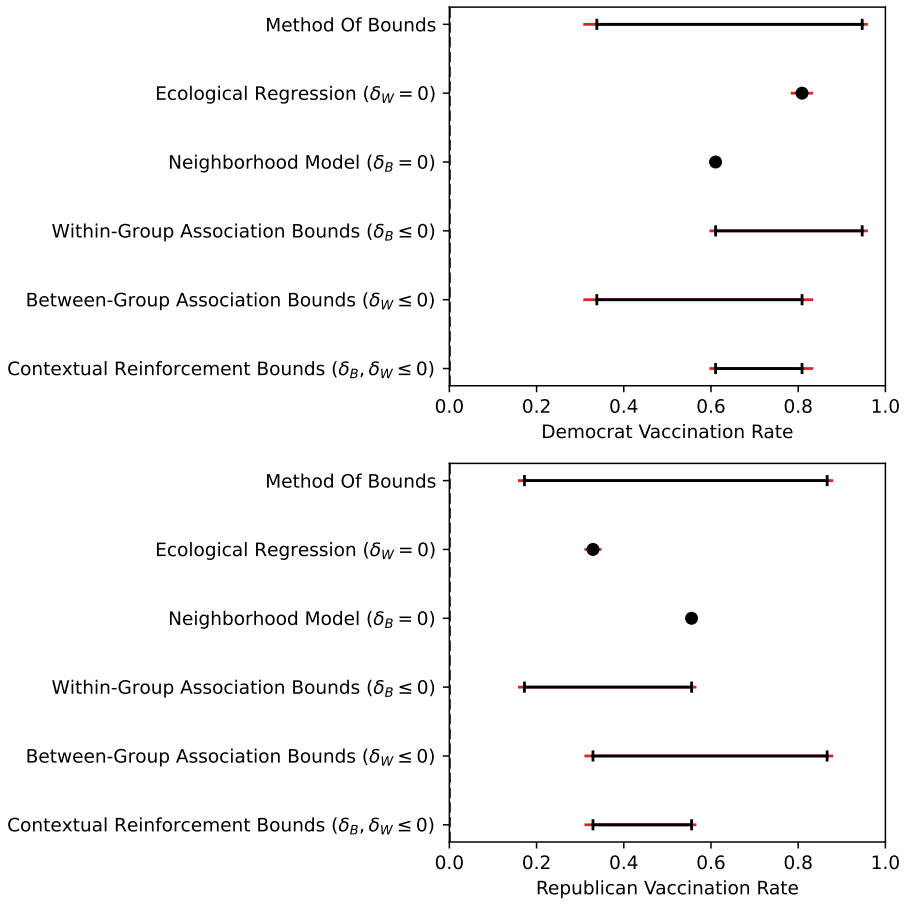
*or (ii)* $\mu'(X_n) \leq 0$, *and*

$$\begin{aligned}
\tilde{D}_n &\in \left[\max\{\mu'(X_n), \tilde{D}^-_{MOB,n}, 0\right], \\
\tilde{Y}^1_n &\in [\max\{\tilde{Y}^{1-}_{MOB,n}, \tilde{Y}_n + (1 - X_n)\mu'(X_n)\}, \tilde{Y}_n] \\
\textit{and } Y^0_n &\in \left[\tilde{Y}_n, \min\{\tilde{Y}_n - X_n \cdot \mu'(X_n), \tilde{Y}^{0+}_{MOB,n}\}\right]
\end{aligned}$$

*(iii) The bounds in (i) and (ii) are sharp in the absence of further information.*

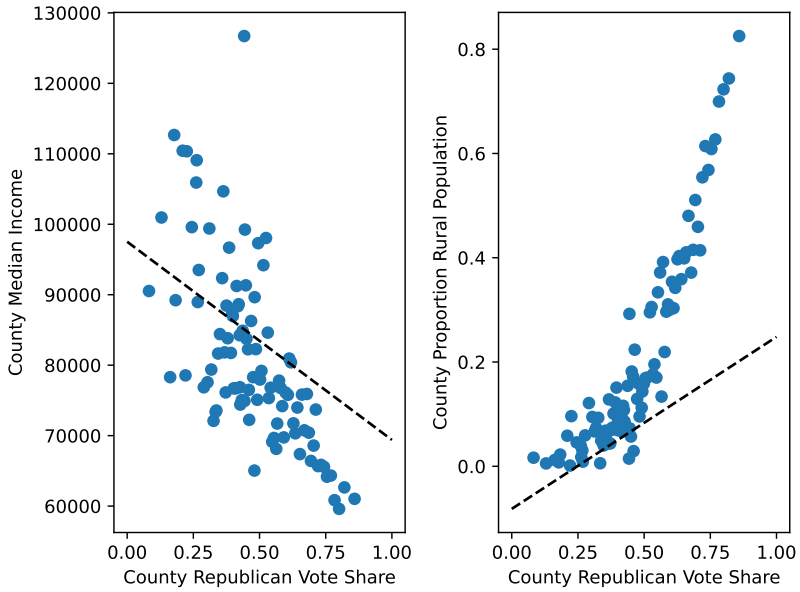
Proofs of these propositions follow similarly to those of the analogous special cases having replaced each quantity with its averaged counterpart.

Figure S1: Identification of Partisan Vaccination Rate



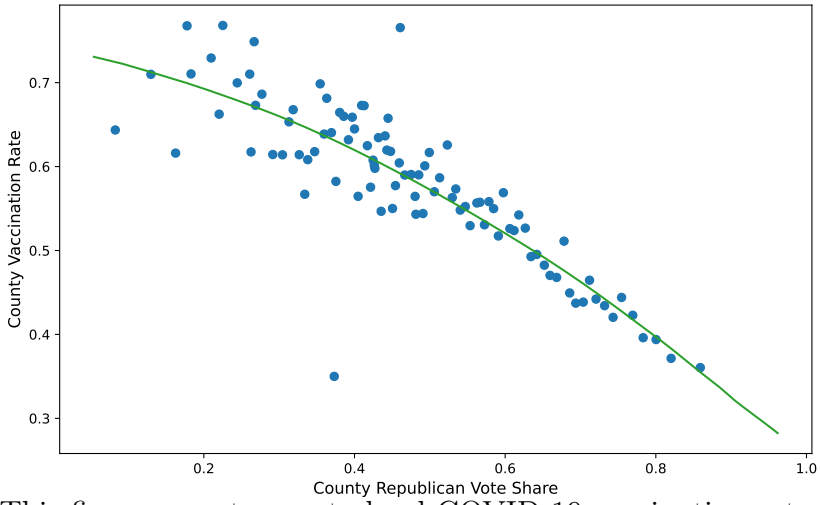

Notes: Mean vaccination rate for Democrats and Republicans (components of the partisan vaccination gap). Red bars are 95% confidence intervals following [Imbens and Manski, 2004]; the confidence intervals are based on standard errors from a county-level bootstrap with 1000 bootstrap replicates.

Figure S2: Median Income and Proportion Rural Population by Republican Vote Share




Notes: The figure reports county-level median income and proportion of the population living in rural areas. Counties are grouped into 100 equal-population bins (blue). Population-weighted linear regressions are also shown (black dotted lines). Household income data and proportion rural population were sourced from the 2023 5-year American Community Survey and the 2020 Decennial Census, respectively. Republican vote share was sourced from the MIT Election Data and Science Lab.

Figure S3: Local Linear Approximation to County Vaccine Data




Notes: This figure reports county-level COVID-19 vaccination rates by the share of voters in the county who voted for the Republican candidate in the 2020 presidential election. Counties are grouped into 100 equal-population bins (blue). The local linear approximation for county-level data is shown in green. This approximation uses an Epanechnikov kernel and 10-fold cross-validation to choose the bandwidth parameter.